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WEAK GLOBAL DIMENSION OF ENDOMORPHISM RINGS OF FREE MODULES

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Let R be a ring, F an $\aleph$-generated free R-module, $\Lambda = \mathrm{End}_R(F)$. If $\aleph$ is finite, then Λ and R are Morita equivalent and so have the same weak and global dimensions. If $\aleph$ is infinite, however, that is no longer true. Indeed, the global dimension of Λ may well depend on the cardinality of the continuum, and is always infinite if $2^{\aleph_0} \geq \aleph_\omega$. The situation for weak dimension is different however.

In the first section of this note, we determine the weak global dimension of Λ when R is $\aleph$-coherent. The main result is a generalization of the well-known facts that, for $\aleph$ infinite,

$$\Lambda \text{ is Von Neumann regular} \Leftrightarrow R \text{ is semi-simple Artinian}$$

and, if R is $\aleph$-noetherian,

$$\Lambda \text{ is semi-hereditary} \Leftrightarrow R \text{ is hereditary.}$$

The results of this section are consequences of the results of Section 2, but the tools used are so elementary that handling this special case first can serve as motivation for the later, more abstract, work.

In the case that R is not $\aleph$-coherent, it is still possible to obtain bounds on the weak global dimension of Λ even though we cannot compute it precisely except in the case that R is perfect. That will occupy the second section of this paper. The crucial theorem of Section 2, interpreted in the category of R-modules, is of definite interest in its own right. Let M be an R-module,

$$0 \longrightarrow K \xrightarrow{j} F \longrightarrow M \longrightarrow 0$$

an exact sequence with F free. It is well known that the following are equivalent:

(i) M is flat.

(ii) For all finite $S \subseteq K$, there is a $\mu : F \to K$ such that $\mu \cdot j(s) = s$ for all $s \in S$.

We also have the equivalence of:

(i′) M is projective.

(ii′) For all $S \subseteq K$, there is a $\mu : F \to K$ such that $\mu \cdot j(s) = s$ for all $s \in S$.
In between the two conditions (ii) and (ii′) are the conditions:

(ii″)$_\aleph$ For all $S \subseteq K$ with $|S| \le \aleph$, there is a $\mu : F \to K$ with $\mu \cdot j(s) = s$ for all $s \in S$.
If we let P be an $\aleph$-generated free module and $\Lambda = \mathrm{End}_R(P)$, condition (ii″)$_\aleph$ is equivalent to the statement that $\mathrm{Hom}_R({}_\Lambda P, M)$ is Λ-flat. Thus there is a whole range of conditions between flatness and projectivity defined by the mapping property (ii″)$_\aleph$, and examples show that distinct cardinals may give rise to distinct classes of modules for which (ii″)$_\aleph$ is satisfied.

Recall that the projective dimension (weak dimension) of a unital module M_R for R a ring with 1 is the smallest n such that there exists an exact sequence

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

where F_i is projective (flat) for all i, or ∞ if no such n exists. We will denote these dimensions by $\mathrm{pd}_R(M)$ (respectively $\mathrm{wd}_R(M)$). The (weak) global dimension of R, denoted (w.)gl.d(R), is the supremum of the (weak) dimensions of all right R-modules.

Standard results on tensor products and flatness show that

$$\begin{aligned} \text{w.gl.d}(R) &= \sup\{\mathrm{wd}(R/I) \mid I \text{ a finitely generated right ideal}\} \\ &= 0 \quad \text{or } 1 + \sup\{\mathrm{wd}(I) \mid I \text{ a finitely generated right ideal}\}. \end{aligned}$$

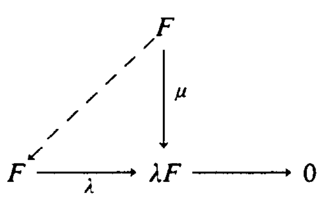
Projective modules are always flat and if a module is finitely presented and flat, it is projective.

1. R $\aleph$-coherent

By our preliminary remarks, we need only look at the finitely generated right ideals of a ring to determine its weak global dimension. Let R, F, and Λ be as in the first sentence of this paper. We say that R is $\aleph$-coherent if, for any map $\alpha : F \to R$, $\ker \alpha$ is $\aleph$-generated (i.e. $\aleph$-generated ideals are $\aleph$-related). A standard argument shows that any $\aleph$-generated submodule of F is $\aleph$-related. These definitions and comments are valid when $\aleph = \aleph_{-1}$ represents the set of all finite cardinals, but we will only be interested in the case where $\aleph$ is infinite. We then have

(*) *Let $\lambda \in \Lambda$. Then there exists a $\mu \in \Lambda$ such that $\mu F = \ker \lambda$.*
We make some additional trivial observations.

(**) *For $\lambda \in \Lambda$, $\lambda\Lambda = \{\mu \in \Lambda \mid \mu F \subseteq \lambda F\}$.*
(**) is an immediate consequence of the projectivity of F, since if $\mu F \subseteq \lambda F$ we have



and so $\mu \in \lambda\Lambda$. Clearly $\mu \in \lambda\Lambda \Rightarrow \mu F \subseteq \lambda F$. (*) and (**) together yield

(***) *Let $\lambda \in \Lambda$. Then there exists a $\mu \in \Lambda$ and an exact sequence*

$$0 \longrightarrow \mu\Lambda \longrightarrow \Lambda \longrightarrow \lambda\Lambda \longrightarrow 0.$$

Proof. (***) is a corollary of (*) and the exactness of Hom(F,). Map $\Lambda \to \lambda\Lambda$ by $\nu \to \lambda\nu$. The kernel of this map is $\{\nu \in \Lambda \mid \nu F \subseteq \ker \lambda\}$. By (*), there exists a μ with $\ker \lambda = \mu F$, and (**) completes the proof.

We next observe that, for any positive integer n, $n \cdot \aleph = \aleph$. Hence $F \approx \bigoplus_1^n F_i$ where each $F_i \approx F$. Then $\bigoplus_1^n \Lambda_i \approx \mathrm{Hom}(\bigoplus_1^n F_i, F) \approx \mathrm{Hom}(F, F) \approx \Lambda$ as right Λ-modules, so finitely generated Λ-modules are cyclic.

We now compute w.gl.d(Λ).

1.1. Theorem. *Let R be $\aleph$-coherent and $\lambda \in \Lambda$. Then*

$$\mathrm{pd}_\Lambda(\lambda\Lambda) = \mathrm{pd}_R(\lambda F).$$

Proof. We use induction on $k = \min\{\mathrm{pd}_\Lambda(\lambda\Lambda), \mathrm{pd}_R(\lambda F)\}$.

If $k = \infty$, we are done.

If $k = 0$, λF is R-projective $\Leftrightarrow$ $F \xrightarrow{\lambda} \lambda F \longrightarrow 0$ is split exact $\Leftrightarrow$ $\ker \lambda$ is a direct summand of F $\Leftrightarrow$ there is an idempotent $e = e^2 \in \Lambda$ such that $\ker \lambda = eF$ $\Leftrightarrow$ there is an exact sequence $0 \to e\Lambda \to \Lambda \to \lambda\Lambda \to 0$ for some idempotent $e = e^2 \in \Lambda$ $\Leftrightarrow$ $\lambda\Lambda \approx (1-e)\Lambda$ for some $e = e^2 \in \Lambda$ $\Leftrightarrow$ $\lambda\Lambda$ is Λ-projective.

If $1 \leq k < \infty$, let $\ker \lambda = \mu F$. Then we have two exact sequences

$$0 \longrightarrow \mu\Lambda \longrightarrow \Lambda \longrightarrow \lambda\Lambda \longrightarrow 0,$$
$$0 \longrightarrow \mu F \longrightarrow F \longrightarrow \lambda F \longrightarrow 0,$$

where the middle term is projective over the appropriate ring. Then

$$\mathrm{pd}_R(\mu F) = \mathrm{pd}_R(\lambda F) - 1 \quad \text{and} \quad \mathrm{pd}_\Lambda(\mu\Lambda) = \mathrm{pd}_\Lambda(\lambda\Lambda) - 1.$$

By induction the theorem is true for all k.

1.2. Corollary. *With the notation of the theorem*

$$\mathrm{w.gl.d}(\Lambda) = \sup\{\mathrm{pd}_R(I) \mid I \textit{ an } \aleph\textit{-generated submodule of } F\} + 1$$
$$(\textit{or } 0 \textit{ if } R \textit{ is semisimple artinian}).$$

Proof. Let I be an $\aleph$-generated submodule of F. Then $I = \lambda F$ for some $\lambda \in \Lambda$ so $\mathrm{pd}_R(I) = \mathrm{pd}_\Lambda(\lambda\Lambda)$. Since $\lambda\Lambda$ has a free resolution

$$\cdots \Lambda \longrightarrow \Lambda \longrightarrow \cdots \xrightarrow{\mu} \Lambda \xrightarrow{\lambda} \lambda\Lambda \longrightarrow 0,$$

the image of each $\Lambda \to \Lambda$ is finitely presented and hence flat $\Leftrightarrow$ projective. Thus $\mathrm{wd}_\Lambda(\lambda\Lambda) = \mathrm{pd}_\Lambda(\lambda\Lambda) = \mathrm{pd}_R(\lambda F)$. This proves the corollary except when the weak global dimension of Λ is 0. But this is true $\Leftrightarrow$ Λ is regular in the sense of Von Neumann $\Leftrightarrow$ for all $\lambda\in\Lambda$, $\lambda F=eF$ for some $e=e^2\in\Lambda$ $\Leftrightarrow$ any $\aleph$-generated submodule of F is a direct summand of F. Since $\aleph\geq\aleph_0$, and R is a submodule of F, this is easily seen to be equivalent to R being semisimple artinian.

1.3. Corollary. *If R is $\aleph$-noetherian, then*

$$\text{w.gl.d}(\Lambda)=\text{gl.d}(R).$$

Remark. For noetherian rings, weak global dimension = global dimension, so in this case Λ and R have the same weak global dimensions. Indeed, we have:

1.4. Corollary. *If R is a commutative noetherian ring with no non-trivial idempotents, then for any non-zero projective module P,*

$$\text{w.gl.d}(R)=\text{w.gl.d}(\text{End}_R(P)).$$

Proof. If P is finitely generated, then trace $P=R$ so $\text{Hom}_R(P,\ )$ is a category equivalence between R-modules and $\text{End}_R(P)$-modules (see [3]). If P is infinitely generated, then P is free (see [2]).

1.5. Corollary. *If $\aleph=\aleph_n$ for some $n\in\omega$ and R is $\aleph$-coherent, then*

$$\text{w.gl.d}(R)\leq\text{w.gl.d}(\Lambda)\leq\text{w.gl.d}(R)+n+1.$$

Proof. The weak dimension of a module is less than or equal to its projective dimension, giving the first inequality. Since the projective dimension of an $\aleph_n$-generated ideal in an $\aleph_n$-coherent ring differs from its weak dimension by at most $n+1$ (see [7]), we get the second inequality.

Both extremes are possible. As mentioned earlier, noetherian rings have $\text{w.gl.d}(R)=\text{w.gl.d}(\Lambda)$, and that is also the case if R is hereditary. For a valuation domain R, if R is not $\aleph_{n-1}$-noetherian, $\text{w.gl.d}(\Lambda)=n+2$ (see [8]). In a Von Neumann regular ring,

$$0=\text{w.gl.d}(R)\leq\text{w.gl.d}(\Lambda)\leq n+1.$$

Here also both ends can occur; the left equality iff R is semisimple artinian, and the right if R is a Boolean algebra generated by an independent set of idempotents of cardinality $\geq\aleph_n$ (see [11]). Of course you can also get all values in between by taking $\aleph_i$-noetherian Boolean algebras, where $i<n$.

2. R not $\aleph$-coherent

In the case that R is not $\aleph$-noetherian (so Λ is not coherent), the above attack fails. We illustrate this by an example.

Let G be the ordered group $\bigoplus_\Omega \mathbb{Z}_\alpha$, where Ω is the first uncountable ordinal and the order is lexicographical. Let F be a field, and $F[X:G]$ the ring of 'polynomials' in an indeterminant X with coefficients in F and exponents non-negative elements of G. Let S be the multiplicatively closed set of $F[X:G]$ consisting of polynomials with non-zero constant term (= coefficient of X^0). Set $T=F[X:G]_S$. Then T is a valuation domain with maximal ideal $\mathfrak{M}$ generated by $\{X^\alpha \mid \alpha \in G, \alpha > 0\}$. Let R be the pullback of the diagram

$$\begin{array}{ccc} & & T \\ & & \downarrow \\ T & \longrightarrow & T/\mathfrak{M} \end{array}$$

namely

$$R=\{(u,v) \mid u,v \in T, u \equiv v \bmod \mathfrak{M}\}.$$

By modifying the proof of Theorem 2.37 of [9] to take account of descending chains in G of order type Ω as well as ω, one can show that $\mathrm{w.gl.d}(R)=2$ and $\mathrm{gl.d}(R)=4$.

2.1. Proposition. *For the above ring R and $F=\bigoplus_1^\infty b_iR$, there exists $\lambda \in \Lambda$ with* $\mathrm{pd}_\Lambda(\lambda\Lambda)=\mathrm{pd}_R(\lambda F)=3$ *but* $\mathrm{wd}_\Lambda(\lambda\Lambda)=2$.

Proof. Let $x=(X^g,0)\in R$ where $0<g\in G$. Let $\lambda:F\to F$, $\lambda(b_0)=b_0x$, $\lambda(b_i)=b_i$ for $i>0$. Then

$$\ker\lambda=\sum_{0<g\in G} b_0(0,X^g)R.$$

For any strictly descending chain

$$\alpha=\{\alpha_0>\alpha_1>\cdots>\alpha_n>\cdots\}$$

of positive elements of G, let $\mu_\alpha\in\Lambda$, $\mu_\alpha(b_i)=b_0(0,X^{\alpha_i})$. Then $\ker\lambda=\varinjlim \mu_\alpha F$ and $0\to\varinjlim\mu_\alpha\Lambda\to\Lambda\to\lambda\Lambda\to 0$ is exact. We also have the exact sequence

$$0\longrightarrow \bigoplus_{i=0}^{\infty}[b_i-b_{i+1}(0,X^{\alpha_i-\alpha_{i+1}})]R\longrightarrow F\longrightarrow \mu_\alpha F\longrightarrow 0$$

and, since the kernel above $\approx F$,

$$0\longrightarrow\Lambda\longrightarrow\Lambda\longrightarrow\mu_\alpha\Lambda\longrightarrow 0$$

is exact.

Since $\mu_\alpha\Lambda$ is not projective, $\mathrm{wd}_\Lambda(\mu_\alpha\Lambda)=1=\mathrm{pd}_\Lambda(\mu_\alpha\Lambda)$. Thus $\mathrm{wd}_\Lambda(\lambda\Lambda)=2$, and $\mathrm{pd}_\Lambda(\varinjlim\mu_\alpha\Lambda)\le 2$ by [7]. The proof used in [8] shows $\mathrm{pd}_\Lambda(\varinjlim\mu_\alpha\Lambda)>1$, so $\mathrm{pd}_\Lambda(\lambda\Lambda)=3$ and also $\mathrm{pd}_R(\lambda F)=3$.

I suspect that the weak global dimension of Λ in the above proposition is 3 which is less than $1+\mathrm{pd}_R(\lambda F)$. Certainly the proposition shows that the proof used for the

Theorem of Section 1 will not work without the coherence hypothesis.

To handle the non-coherent case, we look at the weak dimension of Λ-modules directly, rather than projective dimension. We also work in a categorical setting rather than just in the category of R-modules. We continue to assume that $\aleph$ is infinite.

Let P be an $\aleph$-generated object in an Ab5-category $\mathbf{A}$ such that $P \approx \coprod_{\alpha \in \mathcal{I}} P_\alpha$, where $\aleph = |\mathcal{I}|$ and each $P_\alpha \approx P$. Let Λ denote $\mathbf{A}(P, P)$ as before. A categorical definition of $\aleph$-generated may be found in [10]. For some set Y, let

$$0 \longrightarrow K \xrightarrow{\ j\ } \coprod_{\beta \in Y} P_\beta \xrightarrow{\ f\ } M \longrightarrow 0 \qquad (\#)$$

be exact in $\mathbf{A}$, and set

$$H_X = \mathbf{A}(P, \coprod_X P_\beta) \quad \text{for } X \subseteq Y,$$
$$D = \{X \subseteq Y \mid |X| \leq \aleph\}.$$

D is directed by $\subseteq$, and we have a directed system $\{H_X \mid X \in D\}$ of Λ-submodules of H_Y.

2.2. Lemma. $\mathbf{A}(P, \coprod_Y P_\beta) = \varinjlim_D H_X$.

Proof. P is $\aleph$-generated, so any map from P to a coproduct factors through a subcoproduct on a set of cardinality $\leq \aleph$.

2.3. Lemma. $\mathbf{A}(P, \coprod_Y P_\beta)$ *is Λ-flat.*

Proof. For $X \in D$, $\coprod_X P_\beta \approx P$ by standard infinite set theory. Thus H_X is Λ-flat, and a direct limit of flat modules is flat.

2.4. Theorem. *For the sequence (#), let* $B = \operatorname{coker}(\mathbf{A}(P, K) \to \mathbf{A}(P, \coprod_Y P_\beta))$. *Then B is Λ-flat $\Leftrightarrow$ for all $\sigma : P \to K$, there exists $\tau : \coprod_Y P_\beta \to K$ such that $\tau j \sigma = \sigma$.*

Proof. ($\Rightarrow$) Let B be Λ-flat, $\sigma : P \to K$. Let $\sigma' = j \cdot \sigma : P \to \coprod_Y P_\beta$. Then $\sigma' \in H_X$ for some $X \in D$. We have an isomorphism $\varphi : P \to \coprod_X P_\beta$, and a map $\psi : \coprod_Y P_\beta \to P$, $\psi|_{\coprod_X P_\beta} = \varphi^{-1}$, $\psi|_{\coprod_{Y-X} P_\beta} = 0$. Then $\sigma' = \varphi[\psi\sigma']$ where $\psi\sigma' \in \Lambda$. For $\eta \in \mathbf{A}(P, \coprod_Y P_\beta)$, let $\bar{\eta}$ denote the image of $\mathbf{A}(1_p, f)\eta$ in B. Note that $\bar{\sigma}' = 0$ since σ' factors through j. Since B is Λ-flat, there exist $s_1, \ldots, s_m \in \mathbf{A}(P, \coprod_Y P_\beta)$ and $\lambda_1, \ldots, \lambda_m \in \Lambda$ such that $\bar{\varphi} = \sum_{i=1}^m \bar{s}_i \lambda_i$ and $\lambda_i \psi \sigma' = 0$ for all i. For some $X \in D$, $H_X \supseteq \{s_1, \ldots, s_m\}$ and $H_X = t\Lambda$ for some t. Then $s_i = t\mu_i$ and $\bar{\varphi} = \bar{t}\mu$ where $\mu = \sum_{i=1}^m \mu_i \lambda_i$ and $\mu\psi\sigma' = 0$. Since $\bar{\varphi} - \bar{t}\mu = 0$, $(\varphi - t\mu) = j\tau'$ for some $\tau' : P \to K$, and for $\tau = \tau'\psi$

$$j\tau j\sigma = (\varphi - t\mu)\psi j\sigma = (\varphi\psi\sigma' - t\mu\psi\sigma') = \varphi\psi\sigma' = j\sigma$$

and τ is the required map since j is monic.

($\Leftarrow$) $\mathbf{A}(P, \coprod_Y P_\beta)$ is flat. Since ${}_\Lambda\Lambda \approx \mathbf{A}(P, \coprod_1^n P_i)$, any finitely generated left ideal I of Λ is cyclic, say $I = \Lambda\lambda$. Let

$$ju = \upsilon\lambda \in \mathbf{A}(P, \coprod_Y P_\beta) \cdot I \cap \mathbf{A}(P, K).$$

By hypothesis there is a $\tau : \coprod_Y P_\beta \to K$ such that $\tau j u = u$. Then $\tau v \in \mathbf{A}(P, K)$ and $\mathbf{A}(P, \coprod_Y P_\beta) \cdot I \cap \mathbf{A}(P, K) = \mathbf{A}(P, K) \cdot I$, so by [4], B is flat.

2.5. Corollary. *If M is projective in (#), $\mathbf{A}(P, M)$ is Λ-flat.*

Proof. (#) splits, so $\mathbf{A}(P, M) \approx \operatorname{coker}(\mathbf{A}(P, K) \to \mathbf{A}(P, \coprod_Y P_\beta))$ and the projection of $\coprod_Y P_\beta \to K$ will serve as τ in the theorem.

2.6. Corollary. *If P is projective and generates each of its submodules, then*

$$\text{w.gl.d}(\Lambda) \leq 1 + \sup\{\text{pd}_{\mathbf{A}}(\lambda P) \mid \lambda \in \Lambda\}.$$

Proof. The hypotheses on P insure that it generates each $K \subseteq \coprod_Y P_\beta$. Let

$$\cdots \coprod_{Y_n} P_\beta \longrightarrow \cdots \longrightarrow \coprod_{Y_0} P_\beta \longrightarrow \lambda P \longrightarrow 0 \qquad (\#\#)$$

be a projective resolution of λP. Apply the exact functor $\mathbf{A}(P, \ )$ to get a Λ-flat resolution of $\mathbf{A}(P, \lambda P)$. Since a projective image in (##) will produce a flat image in $\mathbf{A}(P, (\#\#))$,

$$\text{wd}_\Lambda(\mathbf{A}(P, \lambda P)) \leq \text{pd}_{\mathbf{A}}(\lambda P).$$

Any finitely generated ideal of Λ is of the form $\lambda\Lambda \approx \mathbf{A}(P, \lambda P)$ by (**), so standard results in [4] complete the proof.

2.7. Corollary. *If $\mathbf{A}$ is the category of right R-modules and $P \approx \coprod_{\aleph} P$ is a projective generator, then*

$$\text{w.gl.d}(R) \leq \text{w.gl.d}(\Lambda) \leq 1 + \sup\{\text{pd}(\lambda P) \mid \lambda \in \Lambda\}.$$

Proof. The only missing part of the proof is the observation that the mapping back property of the theorem is stronger than a (necessary and) sufficient criterion for flatness found in [4].

2.8. Corollary. *If R is a right perfect ring and $P \approx \coprod_{\aleph} P$ a projective generator, then*

$$\text{w.gl.d}(\Lambda) = \text{gl.d}(R).$$

Proof. For perfect rings R, a module M is flat iff M is projective (see [1]).

It is possible to use the above results to obtain a theorem about R-modules only.

2.9. Theorem. *Let R be any ring, M* a right R-module,

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

a projective resolution of M, $K_n = \ker(P_n \to P_{n-1})$. Let

$$k = \sup\{\text{pd}_R(I) \mid I \textit{ a finitely generated right ideal of } R\}.$$

Then for L any $\aleph_m$-generated submodule of K_{k+m+2}, there exists an R-homomorphism $\tau : P_{k+m+2} \to K_{k+m+2}$ such that $\tau|_L = 1_L$.

Proof. By adding appropriate projectives Q_n to P_n, we may assume each P_n is free. For $F = \bigoplus_{\aleph_m} R_\alpha$ and $\Lambda = \mathrm{End}_R(F)$,

$$\begin{aligned}\text{w.gl.d}(\Lambda) &\le 1 + \sup\{\mathrm{pd}_R(J) \mid J \text{ an } \aleph_m\text{-generated submodule of } F\} \\ &\le 1 + k + m + 1.\end{aligned}$$

Thus $\mathrm{Hom}_R(F, K_{k+m+1})$ is Λ-flat. Apply the previous theorem to the map $F \twoheadrightarrow L \hookrightarrow K_{k+m+2}$.

In the case where $m = -1$ and L is finitely generated, this theorem reduces to a statement that K_k is flat (see [4]).

We also note that, if countably generated right ideals also have projective dimensions bounded by k, all subscripts in the above theorem can be reduced by 1. This is the case, for example, when R is Von Neumann regular (with $k = 0$).

A pair of examples show that the above theorem with the modification for countably generated ideals having dimension $\le k$ is best possible: Let R be a free Boolean algebra on a set of cardinality $\ge \aleph_\omega$, and let $\bar{R}$ be a valuation domain which is not $\aleph_n$-noetherian for any $n \in \omega$. For $n \in \omega$, let I_n be an ideal of R generated by an independent set of idempotents of cardinality $\aleph_n$, and $\bar{I}_n$ an ideal of $\bar{R}$ generated by $\aleph_n$ but no fewer elements. By [11], $\mathrm{pd}_R(I_n) = n$, and by [8], $\mathrm{pd}_{\bar{R}}(\bar{I}_n) = n + 1$. Set $M = \bigoplus_{n=0}^{\infty} R/I_n$ and $\bar{M} = \bigoplus_{n=0}^{\infty} \bar{R}/\bar{I}_n$. For $\aleph = \aleph_m$, $\text{w.gl.d}(\Lambda) = m + 1$ and $\text{w.gl.d}(\bar{\Lambda}) = m + 2$. M and $\bar{M}$ have direct summands which are $\aleph$-resolvable and have projective dimension equal to this weak global dimension. Thus $\mathrm{wd}_\Lambda(\mathrm{Hom}_R(F, M)) = m + 1$ and $\mathrm{wd}_{\bar{\Lambda}}(\mathrm{Hom}_{\bar{R}}(\bar{F}, \bar{M})) = m + 2$, and the mapping back property occurs precisely at K_{m+1} and $\bar{K}_{m+2}$.

A somewhat stronger conclusion is obtained under appropriate coherence hypotheses. The reader is referred to [10] for necessary definitions.

2.10. Theorem. *Let $\aleph$ be an infinite cardinal, and let M be the $\aleph$-union of $\{M_\alpha \mid \alpha \in \mathfrak{A}\}$, where $\mathrm{pd}(M_\alpha) \le n$ for all $\alpha \in \mathfrak{A}$. Let*

$$0 \longrightarrow K_n \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\ f\ } M \longrightarrow 0$$

be exact, where each P_i is a coproduct of $\aleph$-generated projectives. Let N be an $\aleph$-generated subobject of K_n. Then there exists K' with $N \subseteq K' \subseteq K_n$ such that K' is a direct summand of P_n.

Proof. We use finite induction on n. If $n = 0$, Proposition 4.4 of [10] says N is a submodule of a direct summand of P_0 which maps onto some projective M_ν, and we let K' be the kernel of f restricted to that direct summand. If $n > 0$, Corollary 4.5 of [10] shows that the induction hypothesis holds for $\ker(P_0 \to M)$.

This proof completely avoids using endomorphism rings. However I see no way to adapt it to the situation where the M_α may be $\aleph$-generated but require more than $\aleph$ relations.

We conclude with one additional reference to the literature. In [5], Brodskii studies properties of the endomorphism ring Λ of a free R-module F reflecting properties of the ring R. One of his basic tools is the faithfulness and fullness of the functor $\mathrm{Hom}_R(F,\ )$ (see [6]). His general attack is to show that every R-module X has a property $\mathscr{P}$ (e.g. $\mathscr{P}$ = injectivity) iff $\mathrm{Hom}_R(F,X)$ has the property $\mathscr{P}$ over Λ. If $\mathscr{P}$ = projectivity, Brodskii shows that X projective need not imply $\mathrm{Hom}_R(F,X)$ projective, although for $\mathscr{P}$ the property $\aleph$-generated projective, X has $\mathscr{P}$ iff $\mathrm{Hom}_R(F,X)$ has $\mathscr{P}$, where F is $\aleph$-generated free. What we have proved here shows that

$$X \text{ projective} \Rightarrow \mathrm{Hom}_R(F,X) \text{ flat} \Rightarrow X \text{ flat}$$

but in general no arrows can be reversed. Thus the Brodskii approach to relating properties of endomorphism rings of frees to properties of the ring R, which is very powerful when properties relating to injectivity are concerned, does not apply to the properties of projectivity and flatness.

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